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THE LOG-BEHAVIOR OF THE SEQUENCE FOR THE PARTIAL SUM OF A LOG-CONVEX SEQUENCE

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Abstract. In this paper, we study the log-behavior of the partial sum for a log-convex sequence. We prove that the log-convexity of the partial sum for a log-convex sequence is preserved under a mild condition. As applications, we mainly discuss the log-convexity of the partial sums for many famous combinatorial sequences.

1. Introduction

A sequence of positive real numbers $\{z_n\}_{n\geq 0}$ is *log-convex* (or *log-concave*) if $z_n^2 \leq z_{n-1}z_{n+1}$ (or $z_n^2 \geq z_{n-1}z_{n+1}$) for each $n \geq 1$. A log-convex sequence $\{z_n\}_{n\geq 0}$ is *log-balanced* if $\{\frac{z_n}{n!}\}_{n\geq 0}$ is log-concave (Došlić [3] gave the definition of log-balancedness). Log-convexity and log-concavity are not only fertile sources of inequalities, but also play an important role in many fields such as quantum physics, white noise theory, probability, economics and mathematical biology (see the references [1, 2, 4–6, 11, 12, 14]).

It is well known that the binomial coefficients $\binom{n}{k}$, the Eulerian numbers $A(n, k)$, the signless Stirling numbers of the first kind $c(n, k)$ and the Stirling numbers of the second kind $S(n, k)$ are log-concave for k when n is fixed. In combinatorics, there also exist many log-convex sequences. For example, a number of sequences, including the Catalan numbers, the Bell numbers, the Motzkin numbers, the Fine numbers, the Apéry numbers, the large and little Schröder numbers, the derangements numbers, and the central Delannoy numbers, are log-convex. For more log-convex

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combinatorial sequences, see [3,10]. It is obvious that $\{z_n\}_{n\geq 0}$ is log-convex (or log-concave) if and only if its quotient sequence $\{\frac{z_{n+1}}{z_n}\}_{n\geq 0}$ is nondecreasing (nonincreasing) and a log-convex sequence $\{z_n\}$ is log-balanced if and only if $\frac{(n+1)z_n}{z_{n-1}} \geq \frac{nz_{n+1}}{z_n}$ for each $n \geq 1$. Clearly, a log-balanced sequence is log-convex, but its quotient sequence does not grow too fast. The properties of log-convex sequences are different from that of log-concave sequences. For instance, the sequence of the partial sum for a log-concave sequence is still log-concave, but the sequence of the partial sum for a log-convex sequence is not log-convex in general.

This paper focuses on the log-behavior of the partial sum for a log-convex sequence. We give some sufficient conditions for the log-convexity (log-concavity) of the partial sum for a log-convex sequence. As applications of our results, we mainly discuss the log-convexity of the partial sums for many famous combinatorial sequences of numbers such as the Catalan numbers, the Motzkin numbers, the Fine numbers, the large Schröder numbers, the derangements numbers, and the central Delannoy numbers.

2. Main results related to log-behavior of some sequence

In this section, we state and prove the main results of this paper. We first recall two lemmas:

Lemma 2.1. [3] *Let $\{z_n\}_{n\geq 0}$ be a log-convex sequence of positive real numbers defined by the three-term recurrence*

$$z_n = R(n)z_{n-1} + S(n)z_{n-2}, \quad n \geq 2,$$

with given initial conditions z_0 and z_1, where $R(n) \geq 0$ and $S(n) \leq 0$ for $n \geq 2$. For $n \geq 1$, let

$$t_n = \frac{z_n}{z_{n-1}}, \quad \Delta_R(n) = \begin{vmatrix} R(n) & n \\ R(n+1) & n+1 \end{vmatrix} \quad \overline{\Delta}_S(n) = \begin{vmatrix} S(n) & n-1 \\ S(n+1) & n+1 \end{vmatrix}$$

If there is an integer n_0 such that $t_{n_0+1} \leq \frac{n_0+1}{n_0} t_{n_0}$, and if the inequality

$$\Delta_R(n)t_{n-1} + \overline{\Delta}_S(n) \geq 0$$

holds for all $n \geq n_0$, then the sequence $\{z_n\}_{n\geq n_0}$ is log-balanced.

Lemma 2.2. [15] *For a given sequence $\{z_n\}_{n\geq 0}$, if it is log-balanced, then $\{\sqrt{z_n}\}_{n\geq 0}$ is also log-balanced.*

Theorem 2.1. *Let $\{z_n\}_{n\geq 0}$ be a sequence of positive real numbers defined by*

$$z_{n+1} = [1 + f(n)]z_n - f(n)z_{n-1}, \quad n \geq 1, \tag{1}$$

where $f(n) > 0$ for $n \geq 1$.
For $n \geq 0$, let $x_n = \frac{z_{n+1}}{z_n}$. Suppose that the sequence $\{f(n)\}_{n\geq 1}$ is increasing.

(i) *If $\{z_0, z_1, z_2\}$ is log-convex and $x_0 \geq 1$, then $\{z_n\}_{n\geq 0}$ is log-convex.*

(ii) *For $n \geq 2$, let $\Lambda(n) = [(n+1)f(n-1) - nf(n)](x_{n-2} - 1) - f(n) + x_{n-2}$. If $\{z_n\}_{n\geq 0}$ is log-convex with $x_0 \geq 1$ and there exists an integer n_0 such that $x_{n_0} \leq \frac{n_0+1}{n_0} x_{n_0-1}$ and $\Lambda(n) \geq 0$ for $n \geq n_0$, then $\{z_n\}_{n\geq n_0}$ is log-balanced.*

(iii) *If $\{z_0, z_1, z_2\}$ is log-concave, $x_0 \geq 1$ and $x_n \geq f(n+1)$ for $n \geq 0$, then $\{z_n\}_{n\geq 0}$ is log-concave.*

Proof. It is clear that $\{z_n\}_{n\geq 0}$ is log-convex (log-concave) if and only if $\{x_n\}_{n\geq 0}$ is increasing (decreasing) and a log-convex sequence $\{z_n\}_{n\geq 0}$ is log-balanced if and only if $(n+2)x_n - (n+1)x_{n+1} \geq 0$ for $n \geq 0$.

(i) In order to prove the log-convexity of $\{z_n\}_{n\geq 0}$, it is sufficient to show that the sequence $\{x_n\}_{n\geq 0}$ is increasing.

We first prove by induction that $x_n \geq 1$ for each $n \geq 0$. In fact, it is obvious that $x_0 \geq 1$. Next, we assume that $x_n \geq 1$. By (1), we have

$$x_{n'} = 1 + f(n') - \frac{f(n')}{x_{n'-1}}, \quad n' \geq 1. \tag{2}$$

Then we have

$$x_{n+1} - 1 \quad = \quad f(n+1)\left(1 - \frac{1}{x_n}\right) \geq 0.$$

By mathematical induction, $x_n \geq 1$ holds for every $n \geq 0$.

Now we show that the sequence $\{x_n\}_{n\geq 0}$ is increasing by induction. Since $\{z_0, z_1, z_2\}$ is log-convex, we have $x_0 \leq x_1$. For $n \geq 1$, assume that $x_{n-1} \leq x_n$.

Now we prove that $x_n \le x_{n+1}$. By using (2), we obtain

$$\begin{aligned} x_{n+1} - x_n &= f(n+1) - f(n) + \frac{f(n)}{x_{n-1}} - \frac{f(n+1)}{x_n} \\ &= f(n+1) - f(n) + \left(\frac{1}{x_{n-1}} - \frac{1}{x_n}\right) f(n) + \frac{f(n) - f(n+1)}{x_n} \\ &= \left(1 - \frac{1}{x_n}\right)[f(n+1) - f(n)] + \left(\frac{1}{x_{n-1}} - \frac{1}{x_n}\right) f(n). \end{aligned}$$

Noting that $x_n \ge 1$ for $n \ge 0$, $\{f(n)\}_{n\ge1}$ is increasing, and $f(n) \ge 0$, we have $x_{n+1} - x_n > 0$. By mathematical induction, the sequence $\{x_n\}_{n\ge0}$ is increasing. As a result, the sequence $\{z_n\}_{n\ge0}$ is log-convex.

(ii) For $n \ge 1$, let $t_n = \frac{z_n}{z_{n-1}}$, $R(n) = 1 + f(n-1)$, and $S(n) = -f(n-1)$. It is clear that $t_n = x_{n-1}, \Delta_R(n) = 1 + (n+1)f(n-1) - nf(n)$, $\overline{\Delta}_S(n) = (n-1)f(n) - (n+1)f(n-1)$, and

$$\begin{aligned} \Delta_R(n)t_{n-1} + \overline{\Delta}_S(n) &= \Lambda(n) \\ &= [(n+1)f(n-1) - nf(n)](x_{n-2} - 1) - f(n) + x_{n-2}. \end{aligned}$$

It follows from Lemma 2.1 that the sequence $\{z_n\}_{n\ge n_0}$ is log-balanced.

(iii) In a similar way to (i), we can show that $x_n \ge 1$ for each $n \ge 0$. Next we prove by induction that $\{x_n\}_{n\ge0}$ is decreasing. In fact, since $\{z_0, z_1, z_2\}$ is log-concave, it is easy to see that $x_0 \ge x_1$. For $n \ge 1$, assume that $x_{n-1} \ge x_n$. Using (1), we have

$$x_{n+1} - x_n = \frac{(x_n - 1)[f(n+1) - x_n]}{x_n}.$$

Since $x_n \ge 1$ and $x_n - f(n+1) \ge 0$, we have $x_{n+1} - x_n \le 0$. By mathematical induction, we know that $\{x_n\}_{n\ge0}$ is decreasing and hence $\{z_n\}_{n\ge0}$ is log-concave. $\square$

We note that (i) (or (iii)) of Theorem 2.1 is valid for increasing log-convex (log-concave) sequences. For the decreasing log-convex (log-concave) sequences, we have

Theorem 2.2. *Let $\{z_n\}_{n\ge0}$ be a sequence of positive real numbers defined by (1). For $n \ge 0$, let $x_n = \frac{z_{n+1}}{z_n}$.*

(i) If $\{z_0, z_1, z_2\}$ is log-convex, $x_0 < 1$, and $x_n > f(n+1)$ for $n \ge 0$, then $\{z_n\}_{n\ge0}$ is log-convex.

(ii) If $\{z_0, z_1, z_2\}$ *is log-concave,* $x_0 < 1$*, and* $x_n < f(n+1)$ *for* $n \geq 0$*, then* $\{z_n\}_{n\geq 0}$ *is log-concave.*

The proof of Theorem 2.2 is similar to that of Theorem 2.1 (iii) and is omitted here. By applying (i) of Theorem 2.2, we can prove that the sequence $\{\frac{1}{n}\}_{n\geq 1}$ is log-convex. By using (ii) of Theorem 2.2, we can show that the sequence $\{\frac{1}{n!}\}_{n\geq 0}$ is log-concave.

For a log-convex sequence, it is clear that its quotient sequence is increasing and the sequence of its partial sum satisfies the recurrence (1). Now we investigate the log-behavior of the sequence for the partial sum of a log-convex sequence by using Theorem 2.1. We first give the applications for (i)–(ii) of Theorem 2.1.

Corollary 2.1. *For the Catalan numbers* $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ $(n \geq 1)$*, let* $\mathcal{C}_n = \sum_{k=1}^{n} C_k$*. The sequence* $\{\mathcal{C}_n\}_{n\geq 1}$ *is log-balanced.*

Proof. For $n \geq 1$, put $f(n) = \frac{C_{n+1}}{C_n}$, $x_n = \frac{\mathcal{C}_{n+1}}{\mathcal{C}_n}$ and

$$\Lambda(n) = [(n+1)f(n-1) - nf(n)](x_{n-2} - 1) - f(n) + x_{n-2}, \quad (n \geq 3).$$

It is easily to verify that $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\} = \{1, 2, 4, 9, 23, 65\}$ is log-balanced and $x_1 = 2$. Since $\{C_n\}_{n\geq 1}$ is log-convex, $\{f(n)\}_{n\geq 1}$ is increasing. Then the sequence $\{\mathcal{C}_n\}_{n\geq 1}$ is log-convex by (i) of Theorem 2.1.

Since $\{\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\}$ is log-balanced, there exists an integer $n_0 = 5$ such that $x_{n_0} < \frac{n_0+1}{n_0} x_{n_0-1}$. It is easily to see that $f(n) = \frac{2(2n-1)}{n}$ for $n \geq 1$. Then we have

$$\Lambda(n) = \frac{2(2n-4)}{n-1}(x_{n-2} - 1) - \frac{2(2n-1)}{n} + x_{n-2}.$$

Since $\{\mathcal{C}_n\}_{n\geq 1}$ is log-convex, $x_n \geq 2$ for $n \geq 1$. As a result, we have

$$\begin{aligned} \Lambda(n) &\geq 2 + +\frac{2(2n-4)}{n-1} - \frac{2(2n-1)}{n} \\ &= \frac{2(n^2 - 2n - 1)}{(n-1)n} \\ &> 0, \quad (n \geq 5). \end{aligned}$$

It follows from (ii) of Theorem 2.1 that the sequence $\{\mathcal{C}_n\}_{n\geq 5}$ is log-balanced. We note that $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6\}$ is log-balanced. Hence $\{\mathcal{C}_n\}_{n\geq 1}$ is log-balanced. □

For the binomial coefficients $\binom{2n}{n}$ $(n \geq 0)$, let $\mathcal{C}_n^* = \sum_{k=0}^{n} \binom{2k}{k}$. We can show that the sequence $\{\mathcal{C}_n^*\}_{n\geq 0}$ is log-balanced by using (ii) of Theorem 2.1. Its proof is similar to that of Corollary 2.1 and is omitted here.

Example 2.1. The Motzkin numbers $\{M_n\}_{n\geq 0}$ are defined by

$$M_{n+1} = \frac{2n+3}{n+3} M_n + \frac{3n}{n+3} M_{n-1}, \quad n \geq 1, \tag{3}$$

where $M_0 = M_1 = 1$. Some initial values of $\{M_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5
M_n	1	1	2	4	9	21

Corollary 2.2. *For the Motzkin numbers $\{M_n\}_{n\geq 0}$, let*

$$\mathcal{M}_n = \sum_{k=0}^{n} M_k, \quad n \geq 0.$$

Then $\{\mathcal{M}_n\}_{n\geq 0}$ is log-balanced.

Proof. For $n \geq 0$, let $f(n) = \frac{M_{n+1}}{M_n}$, $x_n = \frac{\mathcal{M}_{n+1}}{\mathcal{M}_n}$ and

$$\Lambda(n) = [(n+1)f(n-1) - nf(n)](x_{n-2} - 1) - f(n) + x_{n-2}, \quad (n \geq 2).$$

Došlić [3] showed that $\{M_n\}_{n\geq 0}$ is log-balanced. Then $\{f(n)\}_{n\geq 0}$ is increasing. Noting that $\{\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\} = \{1, 2, 4, 8, 17\}$ is log-balanced and $x_0 = 2$, we have from (i) of Theorem 2.1 that the sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is log-convex.

Since $\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\}$ is log-balanced, there exists an integer $n_0 = 2$ such that $x_{n_0} < \frac{n_0+1}{n_0} x_{n_0-1}$. For an arbitrary real number x, we define

$$g(x) = [(n+1)f(n-1) - nf(n)](x-1) - f(n) + x.$$

Since $\{M_n\}_{n\geq 0}$ is log-balanced, $(n+1)f(n-1) - nf(n) \geq 0$ for $n \geq 1$. This leads to

$$\begin{aligned} g'(x) &= 1 + (n+1)f(n-1) - nf(n) \\ &> 0. \end{aligned}$$

Then the function g is increasing on $(-\infty, +\infty)$. It follows from the log-convexity of $\{\mathcal{M}_n\}_{n\geq 0}$ that $x_n \geq 2$ for $n \geq 0$. This means that

$$\begin{aligned} \Lambda(n) &= g(x_{n-2}) \\ &\geq 2 + (n+1)f(n-1) - (n+1)f(n), \quad n \geq 2. \end{aligned}$$

By means of (3), we obtain

$$\begin{aligned}&2+(n+1)f(n-1)-(n+1)f(n)\\ =\ &2+(n+1)f(n-1)-(n+1)\left[\frac{2n+3}{n+3}+\frac{3n}{(n+3)f(n-1)}\right].\end{aligned}$$

Then we get

$$\begin{aligned}\Lambda(n) &\geq 2+(n+1)f(n-1)-(n+1)\left[\frac{2n+3}{n+3}+\frac{3n}{(n+3)f(n-1)}\right]\\ &= \frac{(n+1)(n+3)f^2(n-1)-(2n^2+3n-3)f(n-1)-3n(n+1)}{(n+3)f(n-1)}.\end{aligned}$$

For an arbitrary real number x, we define

$$h(x)=(n+1)(n+3)x^2-(2n^2+3n-3)x-3n(n+1).$$

We can verify that the function h is increasing on $(-\infty,+\infty)$. On the other hand, Došlić and Veljan [6] proved that

$$f(n-1)\geq \eta_n,\quad n\geq 1,$$

where $\eta_n=\frac{6n}{2n+3}$. We note that

$$h(\eta_n)=\frac{3n(8n^2+9n-9)}{(2n+3)^2}>0\quad (n\geq 2).$$

This implies that

$$\Lambda(n)\geq \frac{h(\eta_n)}{(n+1)f(n-1)}>0.$$

It follows from (ii) of Lemma 2.1 that the sequence $\{\mathcal{M}_n\}_{n\geq 2}$ is log-balanced. We observe that $\{\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3\}$ is also log-balanced. Hence the sequence $\{\mathcal{M}_n\}_{n\geq 0}$ is log-balanced. $\square$

Example 2.2. The Fine numbers $\{f_n\}_{n\geq 0}$ are defined by

$$f_{n+1}=\frac{7n+2}{2n+4}f_n+\frac{2n+1}{n+2}f_{n-1},\quad n\geq 1, \tag{4}$$

where $f_0=1$ and $f_1=0$. Some initial values of $\{f_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6
f_n	1	0	1	2	6	18	57

Corollary 2.3. *For the Fine numbers $\{f_n\}_{n\geq 0}$, let*

$$\mathcal{F}_n=\sum_{k=0}^{n} f_k,\quad n\geq 0.$$

The sequence $\{\mathcal{F}_n\}_{n\geq 0}$ is log-balanced.

Proof. For $n \geq 2$, let $f(n) = \frac{f_{n+1}}{f_n}$, $x_n = \frac{\mathcal{F}_{n+1}}{\mathcal{F}_n}$ and

$$\Lambda(n) = [(n+1)f(n-1) - nf(n)](x_{n-2} - 1) - f(n) + x_{n-2}, \quad (n \geq 4).$$

Došlić [3] proved that $\{f_n\}_{n\geq 2}$ is log-balanced. Then $\{f(n)\}_{n\geq 2}$ is increasing. It is clear that $\{\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4\} = \{2, 4, 10\}$ is log-balanced and $x_2 = 2$. It follows from (i) of Theorem 2.1 that the sequence $\{\mathcal{F}_n\}_{n\geq 2}$ is log-convex. Since $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\} = \{1, 1, 2, 4\}$ is log-convex, $\{\mathcal{F}_n\}_{n\geq 0}$ is log-convex.

Since $\{\mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6\} = \{4, 10, 28, 85\}$ is log-balanced, there exists $n_0 = 4$ such that $x_{n_0} < \frac{n_0+1}{n_0} x_{n_0-1}$. For any real number x, let

$$g(x) = [(n+1)f(n-1) - nf(n)](x-1) - f(n) + x.$$

Since $\{f_n\}_{n\geq 2}$ is log-balanced, $(n+1)f(n-1) - nf(n) \geq 0$ for $n \geq 3$. This means that $g'(x) > 0$. Then the function g is increasing on $(-\infty, +\infty)$. It follows from the log-convexity of $\{\mathcal{F}_n\}_{n\geq 2}$ that $x_n \geq 2$ for $n \geq 2$. This leads to

$$\begin{aligned} \Lambda(n) &= g(x_{n-2}) \\ &\geq 2 + (n+1)f(n-1) - (n+1)f(n), \quad n \geq 4. \end{aligned}$$

By (4), we have

$$f(n) = \frac{7n+2}{2n+4} + \frac{2n+1}{(n+2)f(n-1)}, \quad n \geq 3.$$

Then we obtain

$$\begin{aligned} \Lambda(n) &\geq 2 + (n+1)f(n-1) - (n+1)\left[\frac{7n+2}{2(n+2)} + \frac{2n+1}{(n+2)f(n-1)}\right] \\ &= \frac{2(n+1)(n+2)f^2(n-1) - (7n^2+5n-6)f(n-1) - 2(n+1)(2n+1)}{2(n+2)f(n-1)}. \end{aligned}$$

Liu and Wang [10] showed that

$$f(n-1) \geq \theta_n, \quad n \geq 3,$$

where $\theta_n = \frac{2(2n+3)}{n+3}$. For any real number x, let

$$h(x) = 2(n+1)(n+2)x^2 - (7n^2+5n-6)x - 2(n+1)(2n+1).$$

We can verify that the function h is increasing on $[\theta_n, +\infty)$. By computing, we have

$$h(\theta_n) = \frac{2(8n^3 + 76n^2 + 180n + 117)}{(n+3)^2}.$$

It is obvious that $\Lambda(n) > 0$ for $n \geq 4$. It follows from (ii) of Theorem 2.1 that the sequence $\{\mathcal{F}_n\}_{n\geq 4}$ is log-balanced. We note that $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5\}$ is also log-balanced. Hence $\{\mathcal{F}_n\}_{n\geq 0}$ is log-balanced. □

Example 2.3. Consider the sequence $\{U_n\}_{n\geq 0}$ counting directed animals. The sequence $\{U_n\}_{n\geq 0}$ satisfies

$$(n+1)U_{n+1} = 2(n+1)U_n + 3(n-1)U_{n-1}, \quad n \geq 1, \tag{5}$$

with $U_0 = 1$, $U_1 = 1$, and $U_2 = 2$. This example is Exercise 6.46 of Stanley [13].

Corollary 2.4. *For the sequence $\{U_n\}_{n\geq 1}$ defined by (5), let $\mathcal{U}_n = \sum_{k=0}^{n} U_k$. We have that $\{\mathcal{U}_n\}_{n\geq 0}$ is log-balanced.*

Proof. For $n \geq 0$, put $f(n) = \frac{U_{n+1}}{U_n}$, $x_n = \frac{\mathcal{U}_{n+1}}{\mathcal{U}_n}$ and

$$\Lambda(n) = [(n+1)f(n-1) - nf(n)](x_{n-2} - 1) - f(n) + x_{n-2}, \quad (n \geq 2).$$

Zhang and Zhao [15] proved that $\{U_n^2\}_{n\geq 2}$ is log-balanced. It follows from Lemma 2.2 that $\{U_n\}_{n\geq 2}$ is also log-balanced. On the other hand, $\{U_0, U_1, U_2, U_3\} = \{1, 1, 2, 5\}$ is log-balanced. Therefore the sequence $\{U_n\}_{n\geq 0}$ is log-balanced. Naturally, $\{f(n)\}_{n\geq 0}$ is increasing. We observe that $\{\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\} = \{1, 2, 4, 9\}$ is log-balanced and $x_0 = 2$. Then the sequence $\{\mathcal{U}_n\}_{n\geq 0}$ is log-convex.

In order to prove the log-balancedness of $\{\mathcal{U}_n\}_{n\geq 0}$, we only need to show that $\{\mathcal{U}_n\}_{n\geq 2}$ is log-balanced. Since $\{U_1, U_2, U_3\}$ is log-balanced, there exists an integer $n_0 = 2$ such that $x_{n_0} < \frac{n_0+1}{n_0} x_{n_0-1}$. For any real number x, put

$$g(x) = [(n+1)f(n-1) - nf(n)](x-1) - f(n) + x.$$

Since $\{U_n\}_{n\geq 0}$ is log-balanced, $(n+1)f(n-1) - nf(n) \geq 0$ for $n \geq 1$. This indicates that $g'(x) > 0$. Then the function g is increasing on $(-\infty, +\infty)$. By the log-convexity of $\{\mathcal{U}_n\}_{n\geq 0}$, we derive that $x_n \geq 2$ for every $n \geq 0$. Hence we have

$$\begin{aligned} \Lambda(n) &= g(x_{n-2}) \\ &\geq 2 + (n+1)f(n-1) - (n+1)f(n), \quad n \geq 2. \end{aligned}$$

By applying (5), we derive

$$f(n) = 2 + \frac{3(n-1)}{2(n+1)f(n-1)}, \quad n \geq 1.$$

Then we obtain

$$\begin{aligned} 2 + (n+1)f(n-1) - (n+1)f(n) &= (n+1)f(n-1) - 2n - \frac{3(n-1)}{2(n+1)f(n-1)} \\ &= \frac{2(n+1)^2 f^2(n-1) - 4n(n+1)f(n-1) - 3(n-1)}{2(n+1)f(n-1)}. \end{aligned}$$

For any real number x, put

$$h(x) = 2(n+1)^2 x^2 - 4n(n+1)x - 3(n-1).$$

We can verify that the function h is increasing on $[\frac{n}{n+1}, +\infty)$. Liu and Wang [10] proved that $f(n) \geq \mu_n$, where $\mu_n = \frac{6n}{2n+1}$. Thus we have

$$\begin{aligned} h(\mu_{n-1}) &= \frac{3(n-1)(8n^3 + 12n^2 - 12n - 25)}{(2n-1)^2} \\ &> 0, \quad (n \geq 2). \end{aligned}$$

This implies that $\Lambda(n) \geq 0$ for $n \geq 2$. It follows from (ii) of Theorem 2.1 that the sequence $\{\mathcal{U}_n\}_{n\geq 2}$ is log-balanced. Then $\{\mathcal{U}_n\}_{n\geq 0}$ is log-balanced. □

Zhang and Zhao [15] and Zhao [16] gave some sufficient conditions for log-balancedness of combinatorial sequences. Here (ii) of Theorem 2.1 provides a new sufficient condition for log-balancedness.

Example 2.4. Consider the sequence $\{d_n\}_{n\geq 0}$ of the derangements numbers. The number d_n is equal to the number of n elements with no fixed points and $\{d_n\}$ satisfies the recurrence

$$d_{n+1} = n(d_n + d_{n-1}), \quad n \geq 1,$$

where $d_0 = 1, d_1 = 0$. Some values of $\{d_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6
d_n	1	0	1	2	9	44	265

Corollary 2.5. *For the derangements numbers $\{d_n\}_{n\geq 0}$, let*

$$\mathcal{D}_n^* = \sum_{k=0}^{n} d_k.$$

The sequence $\{\mathcal{D}_n^*\}_{n\geq 0}$ is log-convex.

Proof. For $n \geq 2$, let $f(n) = \frac{d_{n+1}}{d_n}$ and $x_n = \frac{D^*_{n+1}}{D^*_n}$. Liu and Wang [10] proved that $\{d_n\}_{n\geq 2}$ is log-convex. Then $\{f(n\}_{n\geq 2}$ is increasing. We find that $\{\mathcal{D}^*_2, \mathcal{D}^*_3, \mathcal{D}^*_4\} = \{2, 4, 13\}$ is log-convex and $x_2 = 2$. It follows from (i) of Theorem 2.1 that $\{\mathcal{D}^*_n\}_{n\geq 2}$ is log-convex. We note that $\{\mathcal{D}^*_0, \mathcal{D}^*_1, \mathcal{D}^*_2, \mathcal{D}^*_3\} = \{1, 1, 2, 4\}$ is also log-convex. Hence the sequence $\{\mathcal{D}^*_n\}_{n\geq 0}$ is log-convex. □

Example 2.5. The Apéry numbers A_n satisfy

$$A_n = \frac{34n^3 - 51n^2 + 27n - 5}{n^3} A_{n-1} - \frac{(n-1)^3}{n^3} A_{n-2}, \quad n \geq 2,$$

where $A_0 = 1, A_1 = 5$, and $A_2 = 73$.

Corollary 2.6. *For the Apéry numbers $\{A_n\}_{n\geq 0}$, let*

$$\mathcal{A}_n = \sum_{k=0}^{n} A_k.$$

The sequence $\{\mathcal{A}_n\}_{n\geq 0}$ is log-convex.

Example 2.6. The large Schröder number r_n describes the number of lattice paths from $(0, 0)$ to (n, n) and never rising above the line $y = x$. The large Schröder numbers $\{r_n\}_{n\geq 0}$ satisfy

$$r_n = \frac{3(2n-1)}{n+1} r_{n-1} - \frac{n-2}{n+1} r_{n-2}, \quad n \geq 2,$$

where $r_0 = 1$ and $r_1 = 2$.

Corollary 2.7. *For the large Schröder numbers $\{r_n\}_{n\geq 0}$, consider*

$$\mathcal{R}_n = \sum_{k=0}^{n} r_k.$$

The sequence $\{\mathcal{R}_n\}_{n\geq 0}$ is log-convex.

Example 2.7. The central Delannoy number D_n counts the lattice paths from $(0, 0)$ to (n, n) using only the steps $(1, 0)$, $(0, 1)$ and $(1, 1)$. The sequence $\{D_n\}_{n\geq 0}$ satisfies

$$D_n = \frac{3(2n-1)}{n} D_{n-1} - \frac{n-1}{n} D_{n-2}, \quad n \geq 2,$$

where $D_0 = 1$, $D_1 = 3$, $D_2 = 13$ and $D_4 = 63$.

Corollary 2.8. *For the central Delannoy numbers $\{D_n\}_{n\geq 0}$, let*

$$\mathcal{D}_n = \sum_{k=0}^{n} D_k.$$

The sequence $\{\mathcal{D}_n\}_{n\geq 0}$ is log-convex.

Example 2.8. Let $\{T_n\}_{n\geq 1}$ be the sequence of counting directed column-convex polynominoes of height n. The sequence $\{T_n\}_{n\geq 1}$ satisfies

$$T_n = (n+2)T_{n-1} - (n-1)T_{n-2}, \quad n \geq 3, \tag{6}$$

where $T_1 = 1$, $T_2 = 3$, $T_3 = 13$, and $T_4 = 69$. Došlić [3] showed that the sequence $\{T_n\}_{n\geq 1}$ is log-convex.

Corollary 2.9. *For the sequence defined by (6), let*

$$\mathcal{T}_n = \sum_{k=1}^{n} T_k.$$

The sequence $\{\mathcal{T}_n\}_{n\geq 1}$ is log-convex.

Example 2.9. The Bell number B_n (also called exponential number) is the total number of partitions of $\{1, 2, \cdots, n\}$, i.e.,

$$B_n = \sum_{k=0}^{n} S(n, k).$$

Some values of $\{B_n\}_{n\geq 0}$ are as follows:

n	0	1	2	3	4	5	6
B_n	1	1	2	5	15	52	203

Engel [7] first proved that $\{B_n\}_{n\geq 0}$ is log-convex.

Corollary 2.10. *For the Bell numbers $\{B_n\}$, let*

$$\mathcal{B}_n = \sum_{k=0}^{n} B_k.$$

The sequence $\{\mathcal{B}_n\}_{n\geq 0}$ is log-convex.

Example 2.10. Consider the sequence $\{S_n\}_{n\geq 0}$, where S_n denotes the number of ways of placing n labeled balls into n unlabeled (but 2-colored) boxes ($S_0 = 1$). It is clear that $S_n = \sum_{k=1}^{n} 2^k S(n, k)$. Some values of $\{S_n\}_{n\geq 0}$ are as follows:

$$S_0 = 1, \quad S_1 = 1, \quad S_2 = 6, \quad S_3 = 22, \quad S_4 = 94.$$

Liu and Wang [10] showed that $\{S_n\}_{n\geq 0}$ is log-convex.

Corollary 2.11. *For the sequence $\{S_n\}_{n\geq 0}$ counting ways for placing labeled balls into unlabeled (but 2-colored) boxes, let*

$$\mathcal{S}_n = \sum_{k=0}^{n} S_k.$$

The sequence $\{\mathcal{S}_n\}_{n\geq 0}$ is log-convex.

Example 2.11. Consider the sequence $\{h_n\}_{n\geq 0}$, where h_n is the number of the set of all tree-like polyhexes with $n+1$ hexagons. For the properties of $\{h_n\}_{n\geq 0}$, see Harary and Read [9]. The sequence $\{h_n\}_{n\geq 0}$ satisfies

$$(n+1)h_n = 3(2n-1)h_{n-1} - 5(n-2)h_{n-2}, \quad n \geq 2,$$

where $h_0 = h_1 = 1$, $h_2 = 3$, $h_3 = 10$ and $h_4 = 36$. Liu and Wang [10] proved that $\{S_n\}_{n\geq 0}$, $\{h_n\}_{n\geq 0}$ is log-convex.

Corollary 2.12. *For the sequence $\{h_n\}_{n\geq 0}$ counting tree-like polyhexes, let*

$$\mathcal{H}_n^* = \sum_{k=0}^{n} h_k.$$

The sequence $\{\mathcal{H}_n^*\}_{n\geq 0}$ is log-convex.

Example 2.12. Consider the sequence $\{w_n\}_{n\geq 0}$, where w_n counts the number of walks on cubic lattice with n steps, starting and finishing on xy plane and never going below it. For properties of $\{w_n\}_{n\geq 0}$, see Guy [8]. The sequence $\{w_n\}_{n\geq 0}$ satisfies

$$(n+2)w_n = 4(2n+1)w_{n-1} - 12(n-1)w_{n-2}, \quad n \geq 2,$$

where $w_0 = 1, w_1 = 4, w_2 = 17$, $w_3 = 76$ and $w_4 = 456$. Liu and Wang [10] proved that $\{w_n\}_{n\geq 0}$ is log-convex.

Corollary 2.13. *For the sequence $\{w_n\}_{n\geq 0}$ counting walks on cubic lattice, let*

$$\mathcal{W}_n = \sum_{k=0}^{n} w_k.$$

The sequence $\{\mathcal{W}_n\}_{n\geq 2}$ is log-convex.

The proofs of Corollaries 2.6–2.13 are similar to that of Corollary 2.5 and are omitted here.

In the rest of this section, we give the application for (iii) of Theorem 2.1.

Example 2.13. Let r be a positive integer. The n^{th} term of the generalized harmonic numbers $\mathcal{H}_n^{(r)}$ is defined by

$$\mathcal{H}_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.$$

It is clear that $\mathcal{H}_n^{(1)}$ is the classical harmonic number. We can show that the sequence $\{\mathcal{H}_n^{(r)}\}_{n\geq 1}$ is log-concave by the definition of the log-concavity. Here we prove that $\{\mathcal{H}_n^{(r)}\}_{n\geq 1}$ is log-concave by using (iii) of Theorem 2.1.

Corollary 2.14. *The sequence $\{\mathcal{H}_n^{(r)}\}_{n\geq 1}$ is log-concave.*

Proof. For $n \geq 1$, put $f(n) = \frac{n^r}{(n+1)^r}$ and $x_n = \frac{\mathcal{H}_{n+1}^{(r)}}{\mathcal{H}_n^{(r)}}$. It is easily to see that $\{f(n)\}_{n\geq 1}$ is increasing, $f(n) < 1$ and $x_n > 1$. On the other hand, we observe that $\{\mathcal{H}_1^{(r)}, \mathcal{H}_2^{(r)}, \mathcal{H}_3^{(r)}\}$ is log-concave. It follows from (iii) of Theorem 2.1 that the sequence $\{\mathcal{H}_n^{(r)}\}_{n\geq 1}$ is log-concave. □

Example 2.14. It is well known that the Binet form of the Fibonacci sequence $\{F_n\}_{n\geq 0}$ and the Lucas sequence $\{L_n\}_{n\geq 0}$ are

$$F_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + (-1)^n \alpha^{-n},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$.

Using the definition of the log-convexity, we can verify that the sequences $\{F_{2n+1}\}_{n\geq 0}$ and $\{L_{2n}\}_{n\geq 1}$ are log-convex. Now we discuss the log-behavior of the sequence for the partial sum of $\{F_{2n+1}\}_{n\geq 0}$ ($\{L_{2n}\}_{n\geq 1}$).

Corollary 2.15. *For $n \geq 0$, let*

$$\mathcal{F}_n^* = \sum_{k=0}^{n} F_{2k+1} \quad \text{and} \quad \mathcal{L}_n^* = \sum_{k=1}^{n} L_{2k} \quad (n \geq 1).$$

The sequences $\{\mathcal{F}_n^*\}_{n\geq 0}$ and $\{\mathcal{L}_n^*\}_{n\geq 1}$ are log-concave.

Proof. We can prove that the sequences $\{\mathcal{F}_n^*\}_{n\geq 0}$ and $\{\mathcal{L}_n^*\}_{n\geq 1}$ are log-concave by using other method. Here we show that they are log-concave by means of (iii) of Theorem 2.1.

For $n \geq 0$, let $f(n) = \frac{F_{2n+3}}{F_{2n+1}}$ and $x_n = \frac{\mathcal{F}_{n+1}^*}{\mathcal{F}_n^*}$. Since $\{F_{2n+1}\}_{n\geq 0}$ is log-convex, the sequence $\{f(n)\}_{n\geq 0}$ is increasing. It is clear that $\{\mathcal{F}_0^*, \mathcal{F}_1^*, \mathcal{F}_2^*\} = \{1, 3, 8\}$ is log-concave and $x_0 = 3$. One can verify that $\mathcal{F}_n^* = F_{2n+2}$. Then we have

$$\begin{aligned} x_n - f(n+1) &= \frac{F_{2n+4}}{F_{2n+2}} - \frac{F_{2n+5}}{F_{2n+3}} \\ &= \frac{F_{2n+4}F_{2n+3} - F_{2n+2}F_{2n+5}}{F_{2n+2}F_{2n+3}}. \end{aligned}$$

Due to $F_{2n+4}F_{2n+3} - F_{2n+2}F_{2n+5} = 1$, $x_n - f(n+1) > 0$ holds for $n \geq 0$. It follows from (iii) of Theorem 2.1 that the sequence $\{\mathcal{F}_n^*\}_{n\geq 0}$ is log-concave.

Using the similar method, we can show that the sequence $\{\mathcal{L}_n^*\}_{n\geq 1}$ is also log-concave. □

3. Conclusions

We have derived some sufficient conditions for the log-convexity (log-concavity) of the sequence of partial sums for a log-convex sequence. As applications, we mainly discuss the log-convexity of the partial sums for a series of combinatorial sequences. For example, we prove that the sequences of the partial sums for Catalan numbers, Motzkin numbers and Fine numbers are all log-balanced. Our future work is to study the log-behavior of nonlinear recurrence sequences appearing in combinatorics.

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